# When nonnegative matrices are product of nonnegative idempotent matrices?

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**Abstract**

Applications of nonnegative matrices have been of immense interest to both social and physical scientists, particularly to economists and statisticians. This paper considers the question as to when a nonnegative singular matrix can be decomposed as a product of nonnegative idempotents analogous to the well-known result for any arbitrary matrix. It is shown that (i) *all singular nonnegative matrices of rank < 3*, (ii) all singular nonnegative matrices that have a nonnegative von Neumann inverse, (iii) $(0-1)$ nonnegative definite matrices and (iv) periodic matrices, have the property that they decompose into a product of nonnegative idempotents. An example is given that, in general, this need not be true for singular matrices of rank 3 or higher, including stochastic or symmetric matrices. Besides computational techniques, a recent result that a singular nonnegative quasi-permutation matrix is a product of nonnegative idempotents plays a key role in the proofs of the results.

## 1 Introduction

It is well-known that singular matrices with coefficients in a field can be presented as a product of idempotent matrices (cf.Erdos [8]). Different generalizations have been obtained for matrices with entries in noncommutative division rings or other specific types of rings (cf.Laffey, O'Meara-Hanna, Alahmadi-Jain-Lam-Leroy [2], [15], [4]). Nonnegative matrices have found applications in a number of areas including Statistics and Economics. The reader is directed to the classic books on nonnegative matrices (see, for example, Chapter 7 in Bapat-Raghavan [5], Chapter 9 in Berman-Plemmons [7] on Leontief models in Economics, and the book of Seneta [16] on stochastic matrices and Markov chains). The importance of applications of nonnegative matrices to both physical and social sciences can hardly be over emphasized. Thus a natural question is to ask if the above stated result of Erdos for the decomposition of an arbitrary matrix into idempotents holds for nonnegative matrices when we ask that idempotents are also nonnegative. In other words , is it true that a nonnegative singular matrix can be represented as a product of nonnegative idempotent matrices? Recent results tend towards a positive answer. For example it was shown shown recently in [1], that rank one nonnegative real matrices can indeed be decomposed as products of nonnegative real idempotent matrices. In this paper we will show that the rank one and rank two nonnegative real matrices are always product of nonnegative idempotent matrices. The technique is easier than the one used in [1], but we don't get the information about the number of nonnegative idempotent matrices

needed as obtained in [1]. We will also prove that, without any restrictions, any nonnegative matrix $A$ having a nonnegative von Neumann inverse (i.e. a nonegaitive matrix $B$ such that $A = ABA$) is a product of nonnegative idempotents. Finally, we will provide examples of singular (doubly stochastic) matrices of rank 3 that cannot be presented as a product of nonnegative idempotent matrices. We also provide two new families of nonnegative matrices for which the decomposition is possible, namely, (i) the periodic matrices and (ii) the singular nonnegative definite (0-1)-matrices.

## 2 Rank one and two

Throughout the paper $\mathbb{R}^+$ is the set of nonnegative real numbers. Let us first remark that a matrix $A \in M_n(\mathbb{R})$ (resp. in $M_n(\mathbb{R}^+)$) is a product of (resp. nonnegative) idempotent matrices if and only if the same is true for the matrix $PAP^t$ where $P$ is a permutation matrix.

Let us start with the following lemma. Part (a) can be found in our previous paper [3].

**Lemma 1**  (a) If $B \in M_{n \times n}(\mathbb{R}^+)$ is an $n \times n$ matrix which is a product of (resp. nonnegative) idempotents, then the same is true for the matrix $\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix}$ where $C \in M_{n \times 1}(\mathbb{R})$ (resp. $C \in M_{n \times 1}(\mathbb{R}^+)$) and the other blocks elements are of appropriate sizes.

(b) Any strictly upper triangualr matrix $T \in M_n(\mathbb{R})$ (resp. $T \in M_n(\mathbb{R}^+)$) is a product of (resp. nonnegative) idempotent matrices.

(c) If $n > 1$ and a matrix $A \in M_n(\mathbb{R})$ (resp. $A \in M_n(\mathbb{R}^+)$) has only one nonzero row, then it is a product of (resp. nonnegative) idempotent matrices.

(d) Any singular matrix $A \in M_2(\mathbb{R}^+)$ is a product of nonnegative idempotents.

(e) If $A \in M_3(\mathbb{R}^+)$ has its last row and last column zero, then it is a product of nonnegative idempotents.

(f) If $A \in M_n(\mathbb{R})$ (resp. $A \in M_n(\mathbb{R}^+)$), $n \geq 3$, has all its $i^{th}$ rows and columns zero whenever $i \geq 3$, then $A$ is a product of (resp. nonnegative ) idempotent matrices.

**Proof.** (a) This is classic and can be deduced from the identity:

$$\begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & C \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix},$$

where $I_{n-1}$ is the $n - 1 \times n - 1$ identity matrix.

(b) This is an easy consequence of part (a) above.

(c) If $n = 2$, we write $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. When $a = 0$, $A$ is strictly upper triangular and when $a \neq 0$ we can write

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix}.$$

This is the desired product of (nonnegative) idempotent matrices. The general case ($n > 2$) is then easily obtained using part (a) above and an induction.

(d) If $A \in M_2(\mathbb{R}^+)$ is nonzero of rank 1, we may assume, after permuting the rows and columns if necessary, that its second row $L_2$ is is a nonnegative multiple of its first one $L_1$ and we have $L_2 = \alpha L_1$, where $\alpha \in \mathbb{R}^+$. We then have

$$A = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ 0 \end{pmatrix}$$

The first matrix of the above product is a nonnegative idempotent. The part (c) of this lemma shows that the second matrix is also a product of nonneagtive idempotent matrices.

(e) Part (a) and (d) above show that we only have to consider the case when the $2 \times 2$ top left corner submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $A$ is invertible. Let us first consider the case when $c \neq 0$ then we have

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & ac^{-1} \\ 0 & d & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 0 \end{pmatrix}.$$

The first matrix on the right hand side is a product of nonnegative matrices thanks to part (a) and (d) above, the second factor is a nonnegative idempotent.

Now, if $c = 0$ and $b \neq 0$ we can permute the two first rows and the two first columns and apply what we have just done i.e. when the case when $c \neq 0$. Hence we only have to consider the case when $b$ and $c$ are both zero. We can then write

$$\begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 0 \end{pmatrix}$$

This completes the proof.

(f) This conclusion is a direct consequence of the part (e) and (a) above. ∎

We now consider the case of an $n \times n$ nonnegative matrix of rank one. It is known (cf. [1]) that these matrices are product of nonnegative idempotent matrices. We offer here a shorter proof of this fact.

**Proposition 2** *Let $A \in M_n(\mathbb{R}^+)$, $n > 1$, be a nonnegative matrix of rank 1. Then $A$ is a product of nonnegative idempotent matrices.*

**Proof.** Of course, we may assume $A \neq 0$. If $E \in M_n(\mathbb{R}^+)$ is any idempotent matrix and $P$ is a permutation matrix, $PEP^t \in M_n(\mathbb{R}^+)$ is also an idempotent matrix. This shows that we may assume that the first row of the matrix $A$ is nonzero and hence, for $2 \leq j \leq n$, there exist $\alpha_j \in \mathbb{R}^+$ such that $L_j$, the $j^{th}$ row of $A$, is such that $L_j = \alpha_j L_1$. We thus have

$$A = \begin{pmatrix} L_1 \\ L_2 \\ \dots \\ L_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \alpha_2 & 0 & \dots & 0 \\ \alpha_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \alpha_n & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ 0 \dots 0 \\ \dots \\ 0 \dots 0 \end{pmatrix}.$$

The first matrix on the right hand side is nonnegative a idempotent matrix and the last matrix is a product of nonnegative idempotents according to the above lemma 1 (c). ∎

Let us now turn to the case of rank two nonnegative matrices. The following lemma will be useful.

**Lemma 3** *Let $W \subset \mathbb{R}^n$ be a real vector space of dimension two. Then for any finite subset $S \subset (\mathbb{R}^+)^n$ of $W$ with cardinality at least two, there always exist two elements $s_1, s_2 \in S$ such that all the vectors of $S$ are positive combinations of $s_1$ and $s_2$.*

**Proof.** We use an induction on the cardinality of $S$, denoted by $|S|$. If $|S| = 2$ there is nothing to prove. So assume the result holds for such a subset $S$ with $|S| = n$ and let us show that it is also true for $S \cup \{s\}$. We know that there exist $s_1, s_2 \in S$ such that for every $t \in S$ we can write $t = \alpha_t s_1 + \beta_t s_2$, where $\alpha_t, \beta_t$ are nonnegative real numbers. Since $W$ is of dimension two there exists a linear relation between $s_1, s_2$ and $s$. In fact, it easily seen that we can always write one of these three vectors as a nonnegative linear combination

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of the others. If $s$ is a nonnegative linear combination of $s_1$ and $s_2$ we conclude that the elements $s_1$ and $s_2$ satisfy the conditions of the lemma for $S \cup \{s\}$. If $s_1$ is a nonnegative linear combination of $s$ and $s_2$, say $s_1 = \lambda s + \mu s_2$, the pair $s, s_2$ is such that for any $t \in S$, we have $t = \alpha_t s_1 + \beta_t s_2 = \alpha_t(\lambda s + \mu s_2) + \beta_t s_2 = \alpha_t \lambda s + (\alpha_t \mu + \beta_t) s_2$ and the elements $s$ and $s_2$ will satisfy the conditions of the lemma. The last case, when $s_2$ is expressed as a nonnegative linear combination of $s$ and $s_1$ is obtained in the same way. ∎

**Theorem 4** *Let $A \in M_n(\mathbb{R})^+$, $n > 2$, be a nonnegative singular matrix of rank $2$. Then $A$ is a product of nonnegative idempotent matrices.*

**Proof.** Conjugating the given matrix $A$ by a permutation matrix, we see that we may permute the rows as well as columns of $A$ and so assume that the first two rows of $A$ are linearly independent and generate all the other rows. Thanks to the above lemma 3 we may assume that for any $i \geq 3$ there exist $\alpha_i, \beta_i \in \mathbb{R}^+$ such that $L_i = \alpha_i L_1 + \beta_i L_2$. We can thus write

$$A = \begin{pmatrix} L_1 \\ L_2 \\ \dots \\ L_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \alpha_1 & \beta_1 & 0 & \dots & 0 \\ \dots & \dots & 0 & \dots & 0 \\ \alpha_n & \beta_n & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

The first matrix of this last product is already a nonnegative idempotent matrix. Considering the columns $(C_1, C_2, \dots, C_n)$ of the second factor and making use of Lemma 3 we know that there exist $1 \leq i < j \leq n$ such that the columns of this matrix are nonnegative linear combinations of the $C_i$ and $C_j$. Let us write, for every $1 \leq k \leq n$, $C_k = \lambda_k C_i + \mu_k C_j$ with $\lambda_k, \mu_k \in \mathbb{R}^+$. We then have

$$\begin{pmatrix} L_1 \\ L_2 \\ 0 \\ \dots \\ 0 \end{pmatrix} = (C_1, C_2, \dots, C_n) = (C_i, C_j, 0, \dots, 0)B,$$

where the matrix $B$ has the following shape: its rows are all zeros except the first two rows. The first row is $(\lambda_1, \lambda_2, \dots, 1, \lambda_{i+1}, \dots, \lambda_n)$ and the second row of $B$ is $(\mu_1, \mu_2, \dots, 1, \mu_{j+1}, \dots, \mu_n)$. Remarking that the first matrix $(C_i, C_j, 0, \dots, 0)$ have all its rows and columns zero except for the two first ones, the point (f) in Lemma 1 shows that this matrix is a product on nonnegative idempotent matrices. We now start an induction on $n \geq 3$. If $n = 3$ and if $C_3$ is a nonnegative linear combination of $C_1$ and $C_2$ (this is the case when $i = 1, j = 2$) then $B$ is a nonnegative idempotent. So suppose $n = 3$ and $i = 2, j = 3$. In this case we have

$$B = \begin{pmatrix} \lambda_1 & 1 & 0 \\ \mu_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \lambda_1 & 1 & 0 \\ \mu_1 & 0 & 1 \end{pmatrix}.$$

The second matrix of the right hand side is a nonnegative idempotent matrix and the first one is an upper triangular matrix and hence a product of nonnegative idempotent matrices as shown in Lemma 1(b). Interchanging the two first rows and columns of the matrix $B$ corresponding to case $i = 1, j = 3$ we get the following matrix $C$ and its factorization:

$$C = \begin{pmatrix} \mu_2 & 0 & 1 \\ \lambda_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \lambda_2 & 1 & 0 \\ \mu_2 & 0 & 1 \end{pmatrix}.$$

The last matrix is idempotent and the first matrix is a product of nonnegative idempotent matrices thanks to Lemma 1 (a). This shows that our matrix $B$ is always a product of

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nonnegative idempotent matrices when $n = 3$. This also shows that in fact any $3 \times 3$ matrix of rank two is a product of nonnegative idempotent matrices.

Let us now consider the factorization of an $n \times n$ matrix of rank two when $n \geq 4$. The above discussion shows that it is enough to consider the case of a matrix $B \in M_n(\mathbb{R}^+)$ having its two last rows equal to zero. In fact, this case is an easy consequence of the induction hypothesis and part (a) in Lemma 1. ∎

We now give an example of a nonnegative singular matrix that cannot be written as a product of nonnegative idempotent matrices.

**Example 5** Consider the $4 \times 4$ matrix

$$A_\alpha := \begin{pmatrix} \alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \alpha \\ \alpha & 0 & \alpha & 0 \\ 0 & \alpha & 0 & \alpha \end{pmatrix}, \quad where \ \alpha \in \mathbb{R}^+, \ \alpha \neq 0.$$

If $A_\alpha = E_1 \cdots E_n$ is a product of nonnegative idempotent matrices $E_1, E_2, \ldots, E_n \in M_4(\mathbb{R}^+)$, we get that $A_\alpha = A_\alpha E_n$. For $i = 1, 2, 3, 4$, let us write $(x_i, y_i, z_i, t_i)$ for the $i^{th}$ row of the matrix $E_n$. Equating the entries of the rows on both sides we get, after simplification,

For the first row $1 = x_1 + x_2$, $1 = y_1 + y_2$, $0 = z_1 + z_2$ and $0 = t_1 + t_2$.

For the second row $0 = x_3 + x_4$, $0 = y_3 + y_4$, $1 = z_3 + z_4$ and $1 = t_3 + t_4$

For the third row $1 = x_1 + x_3$, $0 = y_1 + y_3$, $1 = z_1 + z_3$ and $0 = t_1 = t_3$

For the fourth row $0 = x_2 + x_4$, $1 = y_2 + y_4$, $0 = z_2 + z_4$ and $1 = t_2 + t_4$

Since all the real numbers $x_i, y_i, z_i, t_i$ must be nonnegative it is easy to conclude from the above equations that the only solution will be $E_n = Id$. This shows that the matrix $A_\alpha$ does not have a presentation as product of nonnegative idempotent matrices. In fact considering $\alpha = 1/2$ we remark that $A_{1/2}$ is a doubly stochastic matrix. Considering the matrix $A_{1/2} A_{1/2}^T$ we may also conclude that even a nonnegative symmetric stochastic matrix cannot always be presented as the product of nonnegative idempotent matrices.

# 3   Nonnegative von Neumann inverse

Before we begin let us state a theorem that plays a key role in this section. Firstly, we need to give a definition.

**Definition 6** *A square matrix $A \in M_{n,m}(\mathbb{R}^+)$ is a quasi-permutation matrix if each row and each column has at most one nonzero element. A partitioned square matrix $A = (A_{ij})$ with $n^2$ blocks is a quasi-permutation by block matrix if there exists a permutation $\sigma \in S_n$ such that the only nonzero blocks of the partition are $A_{i,\sigma(i)}$. Note that the blocks $A_{ij}$, $i \neq j$, are possibly rectangular blocks but the diagonal blocks $A_{ii}$ are square matrices. Furthermore, if each block $A_{ij}$ is a quasi-permutation matrix then $A$ itself is a quasi-permutation matrix.*

**Theorem 7** *(cf. [2], Theorem 7) Let $A$ be a singular nonnegative quasi-permutation matrix. Then $A$ is a product of nonnegative idempotent matrices.*

It was shown in [3] that a nonnegative singular matrix $A \in M_n(\mathbb{R}^+)$ having a nonnegative von Neumann inverse (i.e. a matrix $B \in M_n(\mathbb{R}^+)$ such that $A = ABA$) can be written as a product of nonnegative idempotent matrices. However, there was an implicit assumption that the block matrices of rank one appearing in $A$ were square matrices. But we will show that this assumption was not necessary.Let us first recall the form of nonnegative matrices having nonnegative von Neumann inverse.

**Proposition 8** *(cf. [11], Theorem 1 and Lemma 2) If a nonnegative square matrix A admits a nonnegative von Neumann inverse X (ie. $A = AXA$), then there exists a permutation matrix $P$ such that $PAP^T$ is of the form*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

*where $C$ and $D$ are nonnegative matrices of suitable sizes and $J$ is a direct sum of matrices of the following three types:*

*(1) $\beta x y^T$ where $x, y$ are positive vectors and $\beta$ is a positive real number.*

*(2)*

$$\begin{bmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & 0 & \cdots & 0 \\ 0 & & \beta_{23} x_2 y_3^T & 0 & \cdots & 0 \\ \vdots & & & \ddots & \cdots & \vdots \\ 0 & & & 0 & \cdots & \beta_{d-1,d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & & & 0 & \cdots & 0 \end{bmatrix}$$

*where for $1 \leq i \leq d$ the vectors $x_i, y_j$ are positive and $\beta_{ij}$ is a positive real number.*

*(3) the zero matrix.*

Notice that the positive vectors $x_i$ and $y_j$ are not necessarily of the same size so that the matrices $\beta_{ij} x_i y_j^T$ are possibly rectangular matrices.

We next prove the following key lemma.

**Lemma 9** *Let $A \in M_{n,m}(\mathbb{R}^+)$ be of rank one. Then there exist $E^2 = E \in M_n(\mathbb{R}^+)$, $F^2 = F \in M_m(\mathbb{R}^+)$ and $M \in M_{n,m}(\mathbb{R}^+)$ a quasi-permutation matrix such that $A = EMF$.*

**Proof.** As in Proposition 2, we may assume that the first row of $A$ is nonzero. We let $\alpha_i \in \mathbb{R}^+$ be such that $L_i = \alpha_i L_1$ for all $i \in \{1, \ldots, n\}$ with $\alpha_1 = 1$. Since $L_1$ is not zero there exist an element $0 \neq a \in \mathbb{R}^+$ and $i_0 \in \{1, \ldots, m\}$ such that the $i_0^{th}$ entry of the vector $L_1$ is $a$. We then have $A = EMF$ with $E = \sum_{i=1}^n \alpha_i e_{i1}$, $M = a e_{1 i_0}$ and $F = \sum_{j=1}^m a^{-1} (L_1)_j e_{i_0 j}$. Notice that the matrix $M$ has only one nonzero entry and hence this is indeed a quasi-permutation matrix. ∎

We may now state the analogue of Proposition 11 in the paper [3]. In fact the statement and proof are very similar, the only difference being that the blocks of the permutation by block matrix may be rectangular matrices. The above lemma plays a key role in dealing with rectangular blocks.

**Proposition 10** *Let $A = (A_{ij})$, $1 \leq i, j \leq l$ be a quasi-permutation by blocks matrix associated with the permutation $\sigma \in S_l$. Suppose that, for every $i$, $1 \leq i \leq l$ there exist matrices $E_i, M_i, F_i$ of suitable sizes such that $A_{i\sigma(i)} = E_i M_i F_i$. Then $A$ can be factorized in the following way:*

$$A = diag(E_1, \ldots, E_l)(M) diag(F_{\tau(1)}, \ldots, F_{\tau(l)}) \qquad (*)$$

*where $\tau = \sigma^{-1}$ and the matrix $M = (M_{ij})$ is a quasi-permutation by blocks matrix associated with $\sigma$ and such that $M_{i\sigma(i)} = M_i$.*

*Moreover, if for $1 \leq i \leq n$ the blocks $M_i$ are quasi-permutation matrices, then $M$ is also a quasi-permutation matrix and if $A$ is singular then $A$ is a product of idempotent matrices. In case the matrices $E_i$, $M_i$ and $F_i$ appearing above are all nonnegative then $A$ is a product of nonnegative idempotent matrices.*

**Proof.** Let us call $D$ the matrix on the right hand side of the equality (*) and compute, for $1 \le r, s \le l$, $D_{sr}$ the $(s, r)$ block of $D$.

We have

$$
\begin{aligned}
D_{sr} &= \big(diag(E_1, \ldots, E_l)(M_{ij})diag(F_{\tau(1)}, \ldots, F_{\tau(l)})\big)_{sr} \\
&= \sum_{k=1}^{l} (diag(E_1, \ldots, E_l))_{sk}\big((M_{ij})diag(F_{\tau(1)}, \ldots, F_{\tau(l)})\big)_{kr} \\
&= E_s\big((M_{ij})diag(F_{\tau(1)}, \ldots, F_{\tau(l)})\big)_{sr} \\
&= E_s\big(\sum_t (M_{ij})_{st} diag(F_{\tau(1)}, \ldots, F_{\tau(l)})_{tr}\big) \\
&= E_s M_{s\sigma(s)}(diag(F_{\tau(1)}, \ldots, F_{\tau(l)})_{\sigma(s)r}.
\end{aligned}
$$

If $r \ne \sigma(s)$ we get $(diag(F_{\tau(1)1}, \ldots, F_{\tau(l)l})_{\sigma(s)r} = 0$ and hence $D_{sr} = 0$. If $r = \sigma(s)$ we have $D_{sr} = E_s M_s diag(F_{\tau(1)}, \ldots, F_{\tau(l)})_{\sigma(s)\sigma(s)} = E_s M_s F_{\tau(\sigma(s))} = E_s M_s F_s = A_{s\sigma(s)} = A_{sr}$. This shows that $D_{sr} = A_{sr}$ for all $1 \le r, s \le l$.

For the additional statement we first remark that, as mentioned above a quasi-permutation by block matrix whose blocks are quasi-permutation matrices is itself a quasi-permutation matrix. The result then follows immediately from Lemma 7. ∎

**Theorem 11** *Let $A$ be a singular nonnegative matrix having a nonnegative von Neumann inverse $X$ (i.e. $A = AXA$). Then $A$ is a product of nonnegative idempotent matrices.*

**Proof.** By Proposition 8 $A$ is a quasi-permutation by blocks matrix associated with a permutation $\sigma \in S_l$. We can thus write $A = (A_{ij})$ where the nonzero blocks $A_{i\sigma(i)} \in M_{n_i, n_{\sigma(i)}}(\mathbb{R}^+)$ are matrices of rank 1. Using Lemma 9, we can write $A_{i\sigma(i)} = E_i M_i F_i$ where $E_i$ and $F_i$ are idempotent matrices and $M_i$ is matrix of rank 1. Proposition 10 above shows that our matrix $A$ can be written as $A = diag(E_1, \ldots, E_l) M diag(F_{\tau(1)}, \ldots, F_{\tau(l)})$, where $\tau = \sigma^{-1}$. The two diagonal matrices on the right hand side of this equality are idempotent matrices and the matrix $M$ is a quasi-permutation square matrix. By Theorem 7 the matrix $M$ is a product of nonnegative idempotent matrices, completing the proof. ∎

The following example shows that the condition for a nonnegative matrix to have nonnegative von Neuman inverse is not necessary for the matrix to decompose into a product of nonnegative idempotent matrices.

$$
A := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}
$$

So the nonnegative matrix $A$ can be decomposed as product of nonnegative idempotent matrices. Now let $Z \in M_3(\mathbb{R})$ be such that $A = AZA$. Denote by $x_1, x_2, x_3$ the entries of the first row of $Z$. Comparing the $(1, 1)$ and the $(1, 2)$ entries on both sides of the equality $A = AZA$ we get $x_1 = 1$ and $x_2 = -1$. This shows that none of the von Neumann inverses of $A$ is a nonnegative matrix. The condition of the above theorem is thus only sufficient.

# 4 Periodic nonnegative matrices and 0-1 matrices.

In this last section we will show that a periodic nonnegative matrix is a product of nonegative idempotent matrices with some mild assumption. In particular, in some cases we will ask that the matrix does not have a zero row or a zero column. We will need the notion of Drazin inverse of a real matrix. Let us recall that a Drazin inverse of a matrix $A$ is a matrix $A^D$ such that

$$
A^D A A^D = A^D \quad A A^D = A^D A \quad A^k = A^D A^{k+1},
$$

where $k \in \mathbb{N}$ is the index of $A$, i.e. the smallest positive integer such that $rank(A^k) = rank(A^{k+1})$. Let us remark that if this index $k$ equals 1, we immediately get that $A = AA^D A$, which means that $A^D$ is a von Neuwmann inverse of $A$. Let us also remind that a matrix is periodic if there exist positive integers $s < l$ such that $A^l = A^s$. The following classical lemma is well-known.

**Lemma 12** *Let $A \in M_n(\mathbb{R}^+)$ be a nonnegative matrix with no zero row or zero column such that $A = A_1 + \cdots + A_r$ where the matrices $A_1, \ldots, A_r$ are nonnegative with the property that for every $1 \le i, j \le r$ we have $A_i A_j = 0, i \neq j$. Then there exist a permutation matrix $P$ and square matrices $B_1, \ldots, B_r$ such that*

$$PAP^t = \begin{pmatrix} B_1 & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & B_r \end{pmatrix}$$

*is a diagonal block matrix.*

Let us remark that the above result can be a bit precise: we have for every $i = 1, \ldots, r$ that $PA_iP^t = diag(0, \ldots, B_i, 0, \ldots)$. Furthermore, if every $A_i$ is a product of nonnegative idempotent matrices then the above lemma shows that the matrix $A$ itself is a product of nonnegative idempotent matrices.

**Theorem 13** *Let $A$ be a nonnegative periodic matrix with no zero row or zero column. If either the index of $A$ is 1 or $A > A^n$ for some $n$, then $A$ is a product of nonnegative idempotent matrices.*

**Proof.** Since $A$ is periodic there exist $1 \le s < l$ such that $A^l = A^s$. This shows that $A$ satisfies the polynomial $p(\lambda) = \lambda^s(1 - \lambda^{l-s}) = \lambda^s(1 - \lambda(\lambda^{l-s-1}))$. If we define $q(\lambda) := \lambda^{l-s-1}$ then the Drazin inverse of $A$ is given by $A^D = A^n q(A)^{n+1}$ (cf. Exercise 32 p.173 in [7]) where $n$ is any integer greater or equal to the index of $A$. We remark that this formula is valid for any polynomial satisfied by $A$. Hence we get $A^D$ is a power of $A$, say $A^D = A^k$, where $k = n + (n+1)(l - s - 1)$. This implies that this Drazin inverse is nonnegative. If the index of $A$ is one we have a nonnegative von Neumann inverse and our previous result shows that $A$ is a product of nonnegative idempotent matrices. Now, let us assume that $A \ge A^r$ for some $r > 1$. Then $A \ge A^r = A.A^{r-1} \ge A^r.A^{r-1}$. Continuing this process we get $A \ge A^r A^{m(r-1)}$ for any positive integer $m$ .....(1).

Invoking Goel-Jain (Theorem 3 in [11]), we can write $A = A_1 + A_2 + \cdots + A_m + N$ where each $A_i \ge 0$, $rank A_i = 1$, $A_i A_j = 0$, for $i \neq j$ and $A_i N = 0$ for all $i = 1, 2, ...m$. Moreover $N$ is nilpotent but need not be nonnegative. Our object is to show that $N$ is nonnegative. Since $N = A - A^2 A^D = A - A^t$, where $t = k + 2 = n + 2 + (l - s - 1)(n + 1)$ and $n$ is any integer larger than the index of $A$, we want to find a suitable $t$ such that $A > A^t$. In other words, by (1) we need $t$ to be of the form $r + m(r - 1)$. This implies that we want to find $n \ge \lambda$ where $\lambda$ is the index of $A$ such that $t = n + 2 + (l - s - 1)(n + 1)$ is also of the form $r + m(r - 1)$ for some $m$. Equivalently, we have to find $n \ge \lambda$ and $m$ such that

$$n + 2 + (l - s - 1)(n + 1) = r + m(r - 1)$$

After simplifications this implies that we have to find $n \ge \lambda$ and $m$ such that $(n+1)(l - s) = (m+1)(r-1)$. This is easy. Write $(l-s)(r-1)(\lambda+1) = (l-s)(r\lambda + r - \lambda - 1) = (l-s)(r\lambda + r - \lambda - 2 + 1) = (l-s)(n+1)$ since $r \ge 2$. We then have $n \ge \lambda$ and $(l-s)(n+1) = (m+1(r-1)$ for some $m$, as required. This proves that $N$ is nonnegative.

Since nonnegative nilpotent matrices are always product of nonnegative idempotent matrices (cf. [3]), we conclude that $N$ is a product of nonnegative idempotent matrices. Since

each $A_i'$ is of rank one it is a product of nonnegative idempotents matrices. Now, to finish the proof we invoke the above lemma 12 and conclude that the matrix $A$ is indeed a product of nonnegative idempotent matrices. ■

**Remark 14** *In the case of index one we do have that $A^D$ is a von Neumann inverse and we don't have to use the lemma mentioned above. In particular in this case we don't need to assume that $A$ does not have a zero row or column.*

Let us recall that a matrix $A \in M_n(\mathbb{R})$ is irreducible if there does not exist a permutation matrix $P$ such that $PAP^t$ is of the form $\begin{pmatrix} E & F \\ 0 & G \end{pmatrix}$

In [14] it is shown that if $A \in M_n(\mathbb{R})$ is an irreducible, nonnegative definite 0-1 matrix then $A$ must be the matrix consisting of all entries equal to 1. Such a matrix, being of rank one can be presented in the form of a product of nonnegative idempotent matrices. Since in the same paper [14] it is also shown that for any nonnegative definite 0-1 matrix $A$ there exists a permutation matrix $P$ such that $PAP^t$ is a block diagonal matrix having nonnegative irreducible definite 0-1 matrices as diagonal blocks. Thus we immediately obtain the following proposition.

**Proposition 15** *Let $A \in M_n(\mathbb{R}$ be a singular nonnegative definite 0-1 matrix. Then $A$ is a product of nonnegative idempotent matrices.*

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